SPECTRAL DECOMPOSITION OF AN ELEMENTARY 3-FERMION 2-BODY OPERATOR

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Abstract

The eigenvalues and eigenfunctions of an elementary 3-fermion 2-body operator $3P_g^2 \wedge I^1 \equiv A^3 \sum_{1\le i<j\le 3} P_g^2(i,j)A^3$ acting on a 3-particle antisymmetric finite dimensional Hilbert space have been found. Here P_g^2 denotes the projection operator onto a 2-particle antisymmetric function g^2, while A^3 denotes the 3-particle antisymmetrizing operator.

keywords: spectral decomposition of operators, fermion 2–body operators

1 Introduction

The spectral decomposition of operators is an interesting subject in its own right. This paper arose while we were searching for new conditions for fermion N-representability [1]–[7], [11, 12]. The new condition for fermion N-representability, the "dual P-condition" [8], requires a knowledge of the maximal eigenvalue of the positive semidefinite operator $\binom{N}{2}P_g^2 \wedge I^{\wedge(N-2)}$ acting on an N-particle antisymmetric Hilbert space $\mathcal{H}^{\wedge N}$ (the N-fold Grassmann product of $\mathcal{H}^1$), where P_g^2 is the projection operator onto a 2-particle antisymmetric function $g^2 \in \mathcal{H}^{\wedge 2}$, and $I^{\wedge(N-2)}$ denotes the identity operator on $\mathcal{H}^{\wedge(N-2)}$. We call this operator an elementary N-fermion 2-body operator ("with 2-body interactions"). So far, we were able to find the spectral decomposition of such an

operator for arbitrary g^2 only for $N = 3$, and this paper contains the results. It is realistic to solve the problem for arbitrary N if g^2 is of a special type, e.g. "extreme geminal" [5], and we will publish those results later. Having the spectral decomposition of an elementary N-fermion 2-body operator it is possible to find the reduced 2-particle density operators corresponding to its eigenfunctions and thus obtain some detailed information about the structure of the convex set $\mathcal{P}_N^2$ consisting of all 2-particle fermion N-representable density operators. Especially, the reduced 2-density operators corresponding to the kernel (null-space) of the operator $\binom{N}{2} P_g^2 \wedge I^{\wedge(N-2)}$ are interesting because they form a face in the set $\mathcal{P}_N^2$ exposed by the operator P_g^2 [11]. We have these results for $N = 3$ (they will be published in a separate paper), and for this purpose we have introduced in this paper in the null-space of the operator $3P_g^2 \wedge I^1$ an orthonormal basis, and give explicitly the projection operator onto the kernel. Theorem 1 contains the spectral decomposition of the operator $3P_g^2 \wedge I^1$, while Theorem 2 gives the projection operator $\mathrm{Ker}\,(3P_g^2 \wedge I^1)$ (upper-case K) onto the null-space $\ker(3P_g^2 \wedge I^1)$ (lower-case k) of this operator.

2 Spectral decomposition

Theorem 1 *Let $\mathcal{H}^1$ be a finite dimensional Hilbert space ($\dim \mathcal{H}^1 = n$), and $\mathcal{H}^{\wedge 2} \equiv \mathcal{H}^1 \wedge \mathcal{H}^1$ denotes the 2-particle antisymmetric space generated by $\mathcal{H}^1$ (the 2-fold Grassmann product of $\mathcal{H}^1$). Let P_g^2 denote the 1-dimensional projection operator onto a 2-particle antisymmetric function $g^2 \in \mathcal{H}^{\wedge 2}$ of 1-rank $r = 2s$ possessing the canonical decomposition $g^2 = \sum_{i=1}^{s} \xi_i |\ 2i-1, 2i\ \rangle$ with $\sum_{i=1}^{s} |\xi_i|^2 = 1$, where $|\ 2i-1, 2i\ \rangle \equiv \sqrt{2}\,\varphi_{2i-1}^1 \wedge \varphi_{2i}^1 \equiv \frac{1}{\sqrt{2}} \det(\varphi_{2i-1}^1, \varphi_{2i}^1)$, $\varphi_i^1 \in \mathcal{H}^1$, is the 2-particle normalized determinant. Let the identity operator I^1 on $\mathcal{H}^1$ possess the decomposition $I^1 = \sum_{i=1}^{r=2s} P_i^1 + \sum_{i=r+1}^{n} P_i^1$, where $P_i^1 \equiv \varphi_i^1 \otimes \overline{\varphi}_i^1 \equiv |\ i\ \rangle\langle\ i\ |$ ($i = 1, \ldots, n$) are 1-dim mutually orthogonal projection operators onto the 1-particle functions $|\ i\ \rangle \equiv \varphi_i^1 \in \mathcal{H}^1$ ($i = 1, \ldots, n$) forming an orthonormal basis in $\mathcal{H}^1$. Then, the 3-particle operator $3P_g^2 \wedge I^1 : \mathcal{H}^{\wedge 3} \longrightarrow \mathcal{H}^{\wedge 3}$ possesses the following spectral decomposition*

$$3P_g^2 \wedge I^1 = \sum_{k=1}^{s=r/2} \left(1 - |\xi_k|^2\right)\left(P_{g_{2k-1}}^3 + P_{g_{2k}}^3\right) + \sum_{l=r+1}^{n} P_{g_l}^3 + 0 \cdot \mathrm{Ker}\,(3P_g^2 \wedge I^1). \tag{1}$$

Here, $P_{g_{2k-1}}^3$, $P_{g_{2k}}^3$ ($k = 1, \ldots, s = r/2$), $P_{g_l}^3$ ($l = r+1, \ldots, n$) are 1-dim projectors onto the following eigenfunctions:

$$g^3_{2k-1} \;=\; \frac{1}{\sqrt{1-|\xi_k|^2}} \sum_{\substack{i=1\\(i\neq k)}}^{s} \xi_i|\; 2i-1, 2i, 2k-1\;\rangle = \sqrt{\frac{3}{1-|\xi_k|^2}}\; g^2 \wedge \varphi^1_{2k-1}, \qquad (k = 1, \ldots, s = \frac{r}{2}) \tag{2}$$

$$g^3_{2k} \;=\; \frac{1}{\sqrt{1-|\xi_k|^2}} \sum_{\substack{i=1\\(i\neq k)}}^{s} \xi_i|\; 2i-1, 2i, 2k\;\rangle = \sqrt{\frac{3}{1-|\xi_k|^2}}\; g^2 \wedge \varphi^1_{2k}, \tag{3}$$

$$g^3_l = \sum_{i=1}^{s} \xi_i|\; 2i-1, 2i, l\;\rangle = \sqrt{3}\; g^2 \wedge \varphi^1_l, \quad (l = r+1, \ldots, n), \tag{4}$$

while $\mathrm{Ker}\,(3P^2_g \wedge I^1)$ *denotes the projection operator onto the null-space* $\ker(3P^2_g \wedge I^1)$ *of the operator* $3P^2_g \wedge I^1$*, which is of dimension* $\binom{n}{3} - n$*. The symbols of the type* $|\; 2i-1, 2i, 2k-1\;\rangle \equiv \sqrt{3!}\,\varphi^1_{2i-1} \wedge \varphi^1_{2i} \wedge \varphi^1_{2k-1}$ *denote the appropriate 3-particle normalized determinants.*

Proof. First, we observe that the functions defined by (2–4) are normalized and all are mutually orthogonal, because the determinants differ from each other in at least one 1-particle function. To prove that they are the eigenfunctions belonging to non-zero eigenvalues of the operator $3P^2_g \wedge I^1$, we express P^2_g and I^1 in the following way:

$$P^2_g = \sum_{i,j=1}^{s=r/2} \xi_i \overline{\xi}_j|\; 2i-1, 2i\;\rangle\langle\; 2j-1, 2j\;|$$

$$I^1 = \sum_{k=1}^{s=r/2} \left(P^1_{2k-1} + P^1_{2k}\right) + \sum_{l=r+1}^{n} P^1_l = \sum_{k=1}^{s} \big(|\; 2k-1\;\rangle\langle\; 2k-1\;| + |\; 2k\;\rangle\langle\; 2k\;|\big) + \sum_{l=r+1}^{n} |\; l\;\rangle\langle\; l\;|.$$

Then,

$$
\begin{aligned}
3P_g^2 \wedge I^1 &= 3\sum_{i,j=1}^{s}\sum_{k=1}^{s}\xi_i\overline{\xi}_j|\ 2i-1,2i\ \rangle\langle\ 2j-1,2j\ |\wedge\big(|\ 2k-1\ \rangle\langle\ 2k-1\ |+|\ 2k\ \rangle\langle\ 2k\ |\big)+\\
&+\ 3\sum_{i,j=1}^{s}\sum_{l=r+1}^{n}\xi_i\overline{\xi}_j|\ 2i-1,2i\ \rangle\langle\ 2j-1,2j\ |\wedge|\ l\ \rangle\langle\ l\ |=\\
&= \sum_{k=1}^{s}\left(1-|\xi_k|^2\right)\left(\frac{1}{1-|\xi_k|^2}\sum_{\substack{i,j=1\\(i,j\neq k)}}^{s}\xi_i\overline{\xi}_j|\ 2i-1,2i,2k-1\ \rangle\langle\ 2j-1,2j,2k-1\ |+\right.\\
&+\left.\frac{1}{1-|\xi_k|^2}\sum_{\substack{i,j=1\\(i,j\neq k)}}^{s}\xi_i\overline{\xi}_j|\ 2i-1,2i,2k\ \rangle\langle\ 2j-1,2j,2k\ |\right)+\\
&+\sum_{l=r+1}^{n}\left(\sum_{i,j=1}^{s}\xi_i\overline{\xi}_j|\ 2i-1,2i,l\ \rangle\langle\ 2j-1,2j,l\ |\right)=\\
&= \sum_{k=1}^{s}\left(1-|\xi_k|^2\right)\Big(|\ g_{2k-1}^3\ \rangle\langle\ g_{2k-1}^3\ |+|\ g_{2k}^3\ \rangle\langle\ g_{2k}^3\ |\Big)+\sum_{l=r+1}^{n}|\ g_l^3\ \rangle\langle\ g_l^3\ |=\\
&= \sum_{k=1}^{s}\left(1-|\xi_k|^2\right)\Big(P_{g_{2k-1}}^3+P_{g_{2k}}^3\Big)+\sum_{l=r+1}^{n}P_{g_l}^3.
\end{aligned}
$$

In the proof we have used the fact that $\sqrt{3}\,|\ 2i-1,2i\ \rangle\wedge|\ k\ \rangle=|\ 2i-1,2i,k\ \rangle\equiv\sqrt{3!}\,\varphi_{2i-1}^1\wedge\varphi_{2i}^1\wedge\varphi_k^1\equiv\det(\varphi_{2i-1}^1,\varphi_{2i}^1,\varphi_k^1)$. The symbol of the type $|\ g_{2k-1}^3\ \rangle\langle\ g_{2k-1}^3\ |\equiv g_{2k-1}^3\otimes\overline{g}_{2k-1}^3\equiv P_{g_{2k-1}}^3$ denotes the projection operator onto the function $g_{2k-1}^3\in\mathcal{H}^{\wedge 3}$, while $|\ 2i-1,2i,k\ \rangle\langle\ 2i-1,2i,k\ |$ is the projection operator onto the determinant function $|\ 2i-1,2i,k\ \rangle\in\mathcal{H}^{\wedge 3}$.

Since the 1-dimensional projectors $P_{g_{2k-1}}^3$, $P_{g_{2k}}^3$, $P_{g_l}^3$ are mutually orthogonal, the obtained above expression

$$
3P_g^2\wedge I^1=\sum_{k=1}^{s}\left(1-|\xi_k|^2\right)\Big(P_{g_{2k-1}}^3+P_{g_{2k}}^3\Big)+\sum_{l=r+1}^{n}P_{g_l}^3
$$

is the spectral resolution of the operator $3P_g^2\wedge I^1$ corresponding to the n non-zero eigenvalues. The orthogonal complement in $\mathcal{H}^{\wedge 3}$ of the subspace spanned by the eigenfunctions g_{2k-1}^3, g_{2k}^3 ($k=1,\ldots,s=r/2$), g_l^3 ($l=r+1,\ldots,n$) is the null-space (kernel) of the operator $3P_g^2\wedge I^1$ corresponding to the eigenvalue zero. The projection operator onto this $\binom{n}{3}-n$ dimensional null-space we denote by $\mathrm{Ker}\,(3P_g^2\wedge I^1)$. Thus, we have obtained resolution (1), and this completes the proof.

Since we are interested in the reduced density operators corresponding to the eigenfunctions belonging to the eigenvalue zero of the operator $3P_g^2\wedge I^1$, we have introduced an orthonormal basis in the null-space $\ker(3P_g^2\wedge I^1)$ and have found the projection operator $\mathrm{Ker}\,(3P_g^2\wedge I^1)$ onto this null-space explicitly.

Theorem 2 *The projection operator* $\mathrm{Ker}\,(3P_g^2 \wedge I^1)$ *onto the null-space of the operator* $3P_g^2 \wedge I^1$ $(g^2 \in \mathcal{H}^{\wedge 2})$ *is a sum of mutually orthogonal projectors*

$$\mathrm{Ker}\,(3P_g^2 \wedge I^1) = K_{0,3}^3 + K_{1,2}^3 + K_{2,1}^3 + K_{3,0}^3, \tag{5}$$

corresponding to the orthogonal decomposition of the 3-particle antisymmetric Hilbert space

$$\mathcal{H}^{\wedge 3} = \widetilde{\mathcal{R}}^{\wedge 3} \oplus \mathcal{R}^1 \wedge \widetilde{\mathcal{R}}^{\wedge 2} \oplus \mathcal{R}^{\wedge 2} \wedge \widetilde{\mathcal{R}}^1 \oplus \mathcal{R}^{\wedge 3}, \tag{6}$$

with $\mathcal{H}^1 = \mathcal{R}^1 \oplus \widetilde{\mathcal{R}}^1$, *where* $\mathcal{R}^1$ *denotes the subspace spanned by the orthonormal basis* $\{\varphi_i^1\}_{i=1}^r$, *and* $\widetilde{\mathcal{R}}^1$ *by* $\{\varphi_i^1\}_{i=r+1}^n$. *The projection operators* $K_{0,3}^3$, $K_{1,2}^3$, $K_{2,1}^3$, $K_{3,0}^3$ *can be expressed in the following form:*

$$\begin{aligned}
K_{0,3}^3 &= \sum_{r+1 \le j_1 < j_2 < j_3 \le n} P^3_{j_1,j_2,j_3}, && (7)\\
K_{1,2}^3 &= \sum_{i=1}^{s} \sum_{r+1 \le j_1 < j_2 \le n} \left(P^3_{2i-1,j_1,j_2} + P^3_{2i,j_1,j_2}\right), && (8)\\
K_{2,1}^3 &= \sum_{1 \le i_1 < i_2 \le s} \sum_{j=r+1}^{n} \left(P^3_{2i_1-1,2i_2-1,j} + P^3_{2i_1,2i_2,j} + P^3_{2i_1-1,2i_2,j} + P^3_{2i_1,2i_2-1,j}\right) + \\
&+ \sum_{l=r+1}^{n} \sum_{m=2}^{s} P^3_{f_{l,m}}, && (9)\\
K_{3,0}^3 &= \sum_{1 \le i_1 < i_2 < i_3 \le s} \Big(P^3_{2i_1-1,2i_2-1,2i_3-1} + P^3_{2i_1-1,2i_2-1,2i_3} + P^3_{2i_1-1,2i_2,2i_3-1} + \\
&+ P^3_{2i_1,2i_2-1,2i_3-1} + P^3_{2i_1-1,2i_2,2i_3} + P^3_{2i_1,2i_2-1,2i_3} + P^3_{2i_1,2i_2,2i_3-1} + P^3_{2i_1,2i_2,2i_3}\Big) + \\
&+ \sum_{k=1}^{s} \sum_{m \in J} \left(P^3_{f_{2k-1,m}} + P^3_{f_{2k,m}}\right), && (10)
\end{aligned}$$

$$J = \begin{cases} \{3,\ldots,s\}, & \mathit{for} \quad k=1 \\ \{2,\ldots,k-1,k+1,\ldots,s\}, & \mathit{for} \quad k=2,\ldots,s. \end{cases}$$

Here, P^3_{ijk} *denotes a projection operator onto the determinant* $\mid ijk\, \rangle$, *while* $P^3_{f_{l,m}}$, $P^3_{f_{2k-1,m}}$, $P^3_{f_{2k,m}}$,

are projection operators onto the following functions respectively:

$$
\begin{aligned}
f^3_{l,m} &= N_{lm}\Big(\sum_{i=1}^{m-1}\xi_i\overline{\xi}_m|\ 2i-1,2i,l\ \rangle - \sum_{i=1}^{m-1}|\xi_i|^2|\ 2m-1,2m,l\ \rangle\Big), \qquad (11)\\
& N_{lm} = \Big(\sum_{i=1}^{m-1}|\xi_i|^2\Big)^{-\frac{1}{2}}\Big(\sum_{i=1}^{m}|\xi_i|^2\Big)^{-\frac{1}{2}},\\
f^3_{2k-1,m} &= N_{km}\Big(\sum_{\substack{i=1\\(i\neq k)}}^{m-1}\xi_i\overline{\xi}_m|\ 2i-1,2i,2k-1\ \rangle - \sum_{\substack{i=1\\(i\neq k)}}^{m-1}|\xi_i|^2|\ 2m-1,2m,2k-1\ \rangle\Big), \qquad (12)\\
f^3_{2k,m} &= N_{km}\Big(\sum_{\substack{i=1\\(i\neq k)}}^{m-1}\xi_i\overline{\xi}_m|\ 2i-1,2i,2k\ \rangle - \sum_{\substack{i=1\\(i\neq k)}}^{m-1}|\xi_i|^2|\ 2m-1,2m,2k\ \rangle\Big), \qquad (13)\\
& N_{km} = \Big(\sum_{\substack{i=1\\(i\neq k)}}^{m-1}|\xi_i|^2\Big)^{-\frac{1}{2}}\Big(\sum_{\substack{i=1\\(i\neq k)}}^{m}|\xi_i|^2\Big)^{-\frac{1}{2}}.
\end{aligned}
$$

Proof. The canonical expansion of $g^2 = \sum_{i=1}^{s=r/2}\xi_i\sqrt{2}\,\varphi^1_{2i-1}\wedge\varphi^1_{2i} \equiv \sum_{i=1}^{s=r/2}\xi_i|\ 2i-1,2i\ \rangle$ determines the decomposition of $\mathcal{H}^1 = \mathcal{R}^1\oplus\widetilde{\mathcal{R}}^1$ and the basis $\{\varphi^1_i\}_{i=1}^n$ in $\mathcal{H}^1$, where the functions $\{\varphi^1_i\}_{i=1}^r$ span the subspace $\mathcal{R}^1$, while $\{\varphi^1_i\}_{i=r+1}^n$ is an orthonormal basis in the orthogonal complement $\widetilde{\mathcal{R}}^1$ of the subspace $\mathcal{R}^1$ in $\mathcal{H}^1$. Correspondingly, the identity operator I^1 on $\mathcal{H}^1$ has the decomposition $I^1 = \sum_{i=1}^r P^1_i + \sum_{i=r+1}^n P^1_i \equiv P^1_{1:r} + \widetilde{P}^1_{1:r}$, which induces the decomposition of the identity operator $I^{\wedge 3}$ on $\mathcal{H}^{\wedge 3}$ onto mutually orthogonal projection operators:

$$
I^{\wedge 3} = \big(P^1_{1:r}+\widetilde{P}^1_{1:r}\big)^{\wedge 3} = \sum_{k=0}^{3}\binom{3}{k}P^{\wedge k}_{1:r}\wedge\widetilde{P}^{\wedge(3-k)}_{1:r} = \widetilde{P}^{\wedge 3}_{1:r} + 3P^1_{1:r}\wedge\widetilde{P}^{\wedge 2}_{1:r} + 3P^{\wedge 2}_{1:r}\wedge\widetilde{P}^1_{1:r} + P^{\wedge 3}_{1:r}, \qquad (14)
$$

to which in turn corresponds the decomposition of the 3-particle antisymmetric space $\mathcal{H}^{\wedge 3}$ onto the mutually orthogonal subspaces:

$$
\mathcal{H}^{\wedge 3} = \widetilde{\mathcal{R}}^{\wedge 3}\oplus\mathcal{R}^1\wedge\widetilde{\mathcal{R}}^{\wedge 2}\oplus\mathcal{R}^{\wedge 2}\wedge\widetilde{\mathcal{R}}^1\oplus\mathcal{R}^{\wedge 3}, \qquad (15)
$$

(for proof of the above formulae see e.g. [9], [10]). The subspaces on the r.h.s. of (15) are spanned by 3-particle determinants which differ between themselves in the number of 1-particle functions belonging to $\mathcal{R}^1$ and $\widetilde{\mathcal{R}}^1$ (e.g. $\widetilde{\mathcal{R}}^{\wedge 3}$ is spanned by $\{|\ j_1,j_2,j_3\ \rangle\}_j$, $\varphi^1_j\in\widetilde{\mathcal{R}}^1$, while $\mathcal{R}^1\wedge\widetilde{\mathcal{R}}^{\wedge 2}$ by $\{|\ i,j_1,j_2\ \rangle\}_{i,j}$, $\varphi^1_i\in\mathcal{R}^1$, $\varphi^1_j\in\widetilde{\mathcal{R}}^1$, etc.).

Now, comparing the spectral decomposition of the operator $3P^2_g\wedge I^1$ (1) with the decomposition of the identity operator $I^{\wedge 3}$ (14) it is possible to find all the mutually orthogonal projection operators $K^3_{0,3}$, $K^3_{1,2}$, $K^3_{2,1}$, $K^3_{3,0}$, which constitute the projection operator onto the kernel $\mathrm{Ker}\,(3P^2_g\wedge I^1)$ (5).

$K^3_{0,3})$ $K^3_{0,3}$ denotes the projection operator onto the kernel of $3P^2_g \wedge I^1$ which is contained in the subspace $\widetilde{\mathcal{R}}^{\wedge 3}$ (there are no 1-particle functions from $\mathcal{R}^1$ in the 3-particle determinants). It is seen from (1) that the whole subspace $\widetilde{\mathcal{R}}^{\wedge 3}$ is in the kernel. Hence, from (14):

$$K^3_{0,3} = \widetilde{P}^{\wedge 3}_{1:r} = \sum_{j_1,j_2,j_3=r+1}^{n} P^1_{j_1} \wedge P^1_{j_2} \wedge P^1_{j_3} = \sum_{r+1 \le j_1 < j_2 < j_3 \le n} P^3_{j_1,j_2,j_3},$$

where P^1_j denotes the projection operator onto $\varphi^1_j \in \widetilde{\mathcal{R}}^1$, while $P^3_{j_1,j_2,j_3}$ is the projection operator onto the 3-particle determinant $\mid j_1, j_2, j_3 \rangle \in \widetilde{\mathcal{R}}^{\wedge 3}$. The dimension of the subspace onto which projects $K^3_{0,3}$: $\dim \operatorname{range} K^3_{0,3} = \binom{n-r}{3} = \binom{n-2s}{3}$.

$K^3_{1,2})$ There are no eigenfunctions in (1) belonging to the non-zero eigenvalues which are built up from determinants containing only one function $\varphi^1_i \in \mathcal{R}^1$. Therefore,

$$\begin{aligned} K^3_{1,2} &= 3P^1_{1:r} \wedge \widetilde{P}^{\wedge 2}_{1:r} = \sum_{i=1}^{r} \sum_{r+1 \le j_1 < j_2 \le n} 3P^1_i \wedge P^2_{j_1,j_2} = \sum_{i=1}^{r} \sum_{r+1 \le j_1 < j_2 \le n} P^3_{i,j_1,j_2} = \\ &= \sum_{i=1}^{s=r/2} \sum_{r+1 \le j_1 < j_2 \le n} \left(P^3_{2i-1,j_1,j_2} + P^3_{2i,j_1,j_2} \right), \end{aligned}$$

and $\dim \operatorname{range} K^3_{1,2} = r\binom{n-r}{2} = 2s\binom{n-2s}{2}$.

$K^3_{2,1})$ First, we decompose the projection operator onto the subspace $\mathcal{R}^{\wedge 2} \wedge \widetilde{\mathcal{R}}^1$ in the following way:

$$\begin{aligned} 3P^{\wedge 2}_{1:r} \wedge \widetilde{P}^1_{1:r} &= 3 \left[\sum_{i_1=1}^{s} \left(P^1_{2i_1-1} + P^1_{2i_1} \right) \right] \wedge \left[\sum_{i_2=1}^{s} \left(P^1_{2i_2-1} + P^1_{2i_2} \right) \right] \wedge \sum_{j=r+1}^{n} P^1_j = \\ &= 3 \left[\sum_{i_1=1}^{s} \sum_{i_2=1}^{s} \left(P^1_{2i_1-1} \wedge P^1_{2i_2-1} + P^1_{2i_1} \wedge P^1_{2i_2} + P^1_{2i_1-1} \wedge P^1_{2i_2} + \right.\right. \\ &\left.\left. + P^1_{2i_1} \wedge P^1_{2i_2-1} \right) \right] \wedge \sum_{j=r+1}^{n} P^1_j = \sum_{1 \le i_1 < i_2 \le s} \sum_{j=r+1}^{n} \left(P^3_{2i_1-1,2i_2-1,j} + P^3_{2i_1,2i_2,j} + \right. \\ &\left. + P^3_{2i_1-1,2i_2,j} + P^3_{2i_1,2i_2-1,j} \right) + \sum_{i=1}^{s} \sum_{j=r+1}^{n} P^3_{2i-1,2i,j}. \end{aligned}$$

Comparing this decomposition with (4) we see that only in the last subspace range $\Big(\sum_{i=1}^{s} \sum_{j=r+1}^{n} P^3_{2i-1,2i,j} \Big)$ are there eigenfunctions g^3_l $(l = r+1, \ldots, n)$ belonging to the eigenvalue different from zero. The dimension of this subspace is $s(n-r) = s(n-2s)$. There are $n-r$ orthonormal functions g^3_l. Hence, there still exists a $s(n-r) - (n-r) = (s-1)(n-2s)$ dimensional subspace belonging to the null-space of $3P^2_g \wedge I^1$. In this subspace we introduce the following

orthonormal basis:

$$f^3_{l,m} \;=\; N_{lm}\bigg(\sum_{i=1}^{m-1}\xi_i\overline{\xi}_m|\ 2i-1,2i,l\ \rangle-\sum_{i=1}^{m-1}|\xi_i|^2|\ 2m-1,2m,l\ \rangle\bigg),$$

$$N_{lm}=\bigg(\sum_{i=1}^{m-1}|\xi_i|^2\bigg)^{-\frac{1}{2}}\bigg(\sum_{i=1}^{m}|\xi_i|^2\bigg)^{-\frac{1}{2}},\quad l=r+1,\ldots,n,\quad m=2,\ldots,s.$$

It can be checked that the functions $\{f^3_{l,m}\}$ $(l=r+1,\ldots,n;\quad m=2,\ldots,s.)$ are:

1° orthogonal to $\{g^3_l\}$ $(l=r+1,\ldots,n)$,

$$\begin{aligned}\langle\ g^3_l\ |\ f^3_{l,m}\ \rangle \;&=\; \sum_{i=1}^{s}\overline{\xi}_i\langle\ 2i-1,2i,l\ |N_{lm}\bigg(\sum_{j=1}^{m-1}\xi_j\overline{\xi}_m|\ 2j-1,2j,l\ \rangle+\\ &-\;\sum_{j=1}^{m-1}|\xi_j|^2|\ 2m-1,2m,l\ \rangle\bigg)=\\ &=\;N_{lm}\bigg(\sum_{i=1}^{m-1}|\xi_i|^2\overline{\xi}_m-\sum_{i=1}^{m-1}|\xi_i|^2\overline{\xi}_m\bigg)=0,\end{aligned}$$

obviously $\langle\ g^3_{l_1}\ |\ f^3_{l_2,m}\ \rangle=0$ if $l_1\neq l_2$;

2° normalized,

$$\begin{aligned}&\langle\ f^3_{l,m}\ |\ f^3_{l,m}\ \rangle=N^2_{lm}\bigg(\sum_{i=1}^{m-1}\overline{\xi}_i\xi_m\langle\ 2i-1,2i,l\ |-\sum_{i=1}^{m-1}|\xi_i|^2\langle\ 2m-1,2m,l\ |\bigg)\\ &\bigg(\sum_{j=1}^{m-1}\xi_j\overline{\xi}_m|\ 2j-1,2j,l\ \rangle-\sum_{j=1}^{m-1}|\xi_j|^2|\ 2m-1,2m,l\ \rangle\bigg)=\\ &=N^2_{lm}\bigg(\sum_{i=1}^{m-1}|\xi_i|^2|\xi_m|^2+\sum_{i=1}^{m-1}|\xi_i|^2\sum_{j=1}^{m-1}|\xi_j|^2\bigg)=\\ &=N^2_{lm}\bigg(\sum_{i=1}^{m-1}|\xi_i|^2\bigg)\bigg(|\xi_m|^2+\sum_{j=1}^{m-1}|\xi_j|^2\bigg)=N^2_{lm}\bigg(\sum_{i=1}^{m-1}|\xi_i|^2\bigg)\bigg(\sum_{j=1}^{m}|\xi_j|^2\bigg)=1;\end{aligned}$$

3° mutually orthogonal,

$$\langle\, f^3_{l,m_1} \mid f^3_{l,m_2} \,\rangle_{m_1<m_2} = N_{lm_1}N_{lm_2}\Bigg(\sum_{i=1}^{m_1-1} \overline{\xi}_i\xi_{m_1}\langle\, 2i-1,2i,l \mid - \sum_{i=1}^{m_1-1} |\xi_i|^2\langle\, 2m_1-1,2m_1,l \mid \Bigg)$$

$$\Bigg(\sum_{j=1}^{m_2-1} \xi_j\overline{\xi}_{m_2}\mid 2j-1,2j,l\,\rangle - \sum_{j=1}^{m_2-1} |\xi_j|^2\mid 2m_2-1,2m_2,l\,\rangle \Bigg) =$$

$$= N_{lm_1}N_{lm_2}\Bigg(\sum_{i=1}^{m_1-1} \overline{\xi}_i\xi_{m_1}\langle\, 2i-1,2i,l \mid - \sum_{i=1}^{m_1-1} |\xi_i|^2\langle\, 2m_1-1,2m_1,l \mid \Bigg)$$

$$\Bigg(\sum_{j=1}^{m_1-1} \xi_j\overline{\xi}_{m_2}\mid 2j-1,2j,l\,\rangle + \sum_{j=m_1}^{m_2-1} \xi_j\overline{\xi}_{m_2}\mid 2j-1,2j,l\,\rangle - \sum_{j=1}^{m_2-1} |\xi_j|^2\mid 2m_2-1,2m_2,l\,\rangle \Bigg) =$$

$$= N_{lm_1}N_{lm_2}\Bigg(\sum_{i=1}^{m_1-1} |\xi_i|^2\xi_{m_1}\overline{\xi}_{m_2} - \sum_{i=1}^{m_1-1} |\xi_i|^2\xi_{m_1}\overline{\xi}_{m_2} \Bigg) = 0,$$

obviously $\langle\, f^3_{l_1,m} \mid f^3_{l_2,m} \,\rangle_{l_1\neq l_2} = 0$.

Thus, the sets $\{f^3_{l,m}\}$ $(l = r+1,\dots,n;\quad m = 2,\dots,s)$ and $\{g^3_l\}$ $(l = r+1,\dots,n)$ span the whole subspace range $\left(\sum_{i=1}^{s}\sum_{j=r+1}^{n} P^3_{2i-1,2i,j}\right)$. Now we can find the dimension of $\ker(3P^2_g \wedge I^1)$ contained in $\mathcal{R}^{\wedge 2} \wedge \widetilde{\mathcal{R}}^1$. It is equal to $\dim \mathrm{range}\, K^3_{2,1} = 4(n-r)\binom{s}{2} + [s(n-r)-(n-r)] = 4(n-2s)\binom{s}{2} + (s-1)(n-2s)$.

$K^3_{3,0})$ $K^3_{3,0}$ is a projection operator onto the $\ker(3P^2_g \wedge I^1)$ which is contained in the subspace $\mathcal{R}^{\wedge 3}$. The dimension of the subspace $\mathcal{R}^{\wedge 3}$ is $\binom{r}{3} = \binom{2s}{3}$, and there are $r = 2s$ orthonormal functions $\{g^3_{2k-1}, g^3_{2k}\}$ $(k = 1,\dots,s)$ belonging to the range $(3P^2_g \wedge I^1)$ in $\mathcal{R}^{\wedge 3}$. Hence, the dimension of the null-space in $\mathcal{R}^{\wedge 3}$ is equal to $\binom{2s}{3} - 2s$. To find the projection operator $K^3_{3,0}$ onto this null-space, we have to decompose properly the projection operator $P^{\wedge 3}_{1:r}$ onto the whole subspace $\mathcal{R}^{\wedge 3}$. Here the calculations are longer than in the previous cases considered above, so we give only the milestones.

$$P^{\wedge 3}_{1:r} = \sum_{i_1,i_2,i_3=1}^{s} \left(P^1_{2i_1-1} + P^1_{2i_1}\right) \wedge \left(P^1_{2i_2-1} + P^1_{2i_2}\right) \wedge \left(P^1_{2i_3-1} + P^1_{2i_3}\right) =$$

$$= \sum_{1\le i_1<i_2<i_3\le s} \Big(P^3_{2i_1-1,2i_2-1,2i_3-1} + P^3_{2i_1-1,2i_2-1,2i_3} + P^3_{2i_1-1,2i_2,2i_3-1} +$$

$$+ P^3_{2i_1,2i_2-1,2i_3-1} + P^3_{2i_1-1,2i_2,2i_3} + P^3_{2i_1,2i_2-1,2i_3} + P^3_{2i_1,2i_2,2i_3-1} + P^3_{2i_1,2i_2,2i_3}\Big) +$$

$$+ \sum_{\substack{i_1,i_2=1\\(i_1\neq i_2)}}^{s} \left(P^3_{2i_1-1,2i_1,2i_2-1} + P^3_{2i_1-1,2i_1,2i_2}\right).$$

Here we have used the facts that:

1°

$$\sum_{i_1,i_2,i_3=1}^{s} P^1_{2i_1-1}\wedge P^1_{2i_2-1}\wedge P^1_{2i_3-1} = \sum_{1\le i_1<i_2<i_3\le s} 3!\, P^1_{2i_1-1}\wedge P^1_{2i_2-1}\wedge P^1_{2i_3-1} = \\ = \sum_{1\le i_1<i_2<i_3\le s} P^3_{2i_1-1,2i_2-1,2i_3-1},$$

and similarly for the projector with three even indices;

2°

$$\sum_{i_1,i_2,i_3=1}^{s} \Big(P^1_{2i_1-1}\wedge P^1_{2i_2-1}\wedge P^1_{2i_3} + P^1_{2i_1-1}\wedge P^1_{2i_2}\wedge P^1_{2i_3-1} + P^1_{2i_1}\wedge P^1_{2i_2-1}\wedge P^1_{2i_3-1}\Big) =$$

$$= \sum_{i_1,i_2=1}^{s} P^1_{2i_1-1}\wedge P^1_{2i_2-1}\wedge P^1_{2i_1} + \sum_{i_1,i_2=1}^{s} P^1_{2i_1-1}\wedge P^1_{2i_2-1}\wedge P^1_{2i_2} +$$

$$+ \sum_{\substack{i_1,i_2,i_3=1\\(i_1\ne i_2\ne i_3)}}^{s} P^1_{2i_1-1}\wedge P^1_{2i_2-1}\wedge P^1_{2i_3} + \sum_{i_1,i_3=1}^{s} P^1_{2i_1-1}\wedge P^1_{2i_1}\wedge P^1_{2i_3-1} +$$

$$+ \sum_{i_1,i_2=1}^{s} P^1_{2i_1-1}\wedge P^1_{2i_2}\wedge P^1_{2i_2-1} + \sum_{\substack{i_1,i_2,i_3=1\\(i_1\ne i_2\ne i_3)}}^{s} P^1_{2i_1-1}\wedge P^1_{2i_2}\wedge P^1_{2i_3-1} +$$

$$+ \sum_{i_1,i_3=1}^{s} P^1_{2i_1}\wedge P^1_{2i_1-1}\wedge P^1_{2i_3-1} + \sum_{i_1,i_2=1}^{s} P^1_{2i_1}\wedge P^1_{2i_2-1}\wedge P^1_{2i_1-1} +$$

$$+ \sum_{\substack{i_1,i_2,i_3=1\\(i_1\ne i_2\ne i_3)}}^{s} P^1_{2i_1}\wedge P^1_{2i_2-1}\wedge P^1_{2i_3-1} = \sum_{\substack{i_1,i_2=1\\(i_1\ne i_2)}}^{s} 3!\, P^1_{2i_1-1}\wedge P^1_{2i_1}\wedge P^1_{2i_2-1} +$$

$$+ \sum_{1\le i_1<i_2<i_3\le s} 3!\,\Big(P^1_{2i_1-1}\wedge P^1_{2i_2-1}\wedge P^1_{2i_3} + P^1_{2i_1-1}\wedge P^1_{2i_2}\wedge P^1_{2i_3-1} + P^1_{2i_1}\wedge P^1_{2i_2-1}\wedge P^1_{2i_3-1}\Big) =$$

$$= \sum_{\substack{i_1,i_2=1\\i_1\ne i_2}}^{s} P^3_{2i_1-1,2i_1,2i_2-1} + \sum_{1\le i_1<i_2<i_3\le s}\Big(P^3_{2i_1-1,2i_2-1,2i_3} + P^3_{2i_1-1,2i_2,2i_3-1} + P^3_{2i_1,2i_2-1,2i_3-1}\Big).$$

Similarly we proceed with the projectors containing two even and one odd indices.

It follows from (1) that only in the subspace range $\sum_{\substack{i_1,i_2=1\\(i_1\ne i_2)}}^{s}\Big(P^3_{2i_1-1,2i_1,2i_2-1} + P^3_{2i_1-1,2i_1,2i_2}\Big)$ are there $r=2s$ eigenfunctions $\{g^3_{2k-1}, g^3_{2k}\}$ $(k=1,\ldots,s=r/2)$ belonging to non-zero eigenvalues of $3P^2_g\wedge I^1$. Since this subspace is $2s(s-1)$ dimensional, there still exists in it the $2s(s-1)-2s = 2s(s-2)$ dimensional null-space of $3P^2_g\wedge I^1$. Within this subspace we can establish the following

orthonormal basis:

$$
\begin{aligned}
f^3_{2k-1,m} &= N_{km}\Bigg(\sum_{\substack{i=1\\(i\neq k)}}^{m-1} \xi_i\overline{\xi}_m|\ 2i-1,2i,2k-1\ \rangle - \sum_{\substack{i=1\\(i\neq k)}}^{m-1} |\xi_i|^2|\ 2m-1,2m,2k-1\ \rangle\Bigg),\\
f^3_{2k,m} &= N_{km}\Bigg(\sum_{\substack{i=1\\(i\neq k)}}^{m-1} \xi_i\overline{\xi}_m|\ 2i-1,2i,2k\ \rangle - \sum_{\substack{i=1\\(i\neq k)}}^{m-1} |\xi_i|^2|\ 2m-1,2m,2k\ \rangle\Bigg),\\
N_{km} &= \Bigg(\sum_{\substack{i=1\\(i\neq k)}}^{m-1} |\xi_i|^2\Bigg)^{-\frac{1}{2}}\Bigg(\sum_{\substack{i=1\\(i\neq k)}}^{m} |\xi_i|^2\Bigg)^{-\frac{1}{2}},\\
& k=1,\ldots,s; \quad m\in J, \quad J=\begin{cases} \{3,\ldots,s\}, & \text{for} \quad k=1\\ \{2,\ldots,k-1,k+1,\ldots,s\}, & \text{for} \quad k=2,\ldots,s. \end{cases}
\end{aligned}
$$

The orthogonality between any function with "odd index k" to any one with "even index k" as well as between functions with different indices k is seen by inspection. The normalization, and other orthogonality relations can be proved in a way similar to that used in the previous case $K^3_{2,1}$, and we omit these calculations for brevity. Now, we can calculate the dimension of the $\ker(3P^2_g \wedge I^1)$ contained in $\mathcal{R}^{\wedge 3}$: it is equal to $\dim \operatorname{range} K^3_{3,0} = 8\binom{s}{3} + 2s(s-2)$. Thus, in principle the proof is done. To make sure that everything that belongs to the $\ker(3P^2_g \wedge I^1)$ is taken into account, we can perform the dimensionality test. The subspace $\mathcal{H}^{\wedge 3}$ is $\binom{n}{3}$ dimensional, there are n orthonormal eigenfunctions belonging to the non-zero eigenvalues of the operator $3P^2_g \wedge I^1$, therefore the dimension of $\ker(3P^2_g \wedge I^1)$ is $\binom{n}{3} - n$. On the other hand this should be equal to

$$
\begin{aligned}
&\sum_{i=0}^{3} \dim \operatorname{range} K^3_{i,3-i} =\\
&\quad = \binom{n-2s}{3} + 2s\binom{n-2s}{2} + \left[4(n-2s)\binom{s}{2} + (s-1)(n-2s)\right] + \left[8\binom{s}{3} + 2s(s-2)\right].
\end{aligned}
$$

This sum is really equal to $\binom{n}{3} - n$, and this completes the proof.

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